AN ACCOUNT OF CAUCHY'S "CALCUL DES RESIDUS."

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(*Continued from page* 9.)

2. Let $\varphi(z)=z^{p-1}$ where $m+1>p>1$. We have then

$$f(\infty+0i)=\infty.$$

But since the value ∞ of x agrees with the upper limit of the x-integration and the value 0 of y with the lower limit of the y-integration, there is no corresponding correction by virtue of (10^{v})*. Since

$$\frac{(\pm\infty+yi)^{p-1}}{(a\pm\infty+yi)^m}=\frac{(x+\infty i)^{p-1}}{(a+x+\infty i)^m}=0,$$

if $m+1>p>1$ condition (11) is fulfilled and we have by formulas $(10'_1)$, $(10'_2)$ and $(10'_3)$

$$\int_{-\infty}^{\infty}\frac{x^{p-1}dx}{(a+x)^{2n}}=\frac{2(-a)^{p-1}}{2n-1}\infty^{2n-1}+\pi i\frac{p-1}{1}\cdot\frac{p-2}{2}\cdots\frac{p-2n+1}{2n-1}\times(-a)^{p-2n}, \qquad (c_1)$$

$$\int_{-\infty}^{\infty}\frac{x^{p-1}dx}{(a+x)^{2n+1}}=\frac{2(p-1)(-a)^{p-2}}{2n-1}\infty^{2n-1}+\pi i\frac{p-1}{1}\cdot\frac{p-2}{2}\cdots\frac{p-2n}{2n}\times(-a)^{p-2n-1}, \qquad (c_2)$$

$$\int_{-\infty}^{\infty}\frac{x^{p-1}dx}{a+x}=\pi i(-a)^{p-1}. \qquad (c_3)$$

In order to separate the real from the imaginary we have to transform these integrals by (13). We have then

$$\int_{-\infty}^{\infty}\frac{x^{p-1}dx}{(a+x)^m}=\int_0^{\infty}x^{p-1}\,dx\left[\frac{1}{(a+x)^m}+\frac{(-1)^{p-1}}{(a-x)^m}\right]$$
$$=\int_0^{\infty}\frac{x^{p-1}dx}{(a^2-x^2)^m}\left[(a-x)^m+e^{(p-1)\pi i}(a+x)^m\right].$$

We have then

$$\int_0^{\infty}\frac{x^{p-1}dx}{(a^2-x^2)^{2n}}\left[(a-x)^{2n}-e^{p\pi i}(a+x)^{2n}\right]=-\frac{2}{2n-1}a^{p-1}e^{p\pi i}\infty^{2n-1}$$
$$+\pi i\frac{p-1}{1}\cdot\frac{p-2}{2}\cdots\frac{p-2n+1}{2n-1}a^{p-2n}e^{p\pi i},$$

consequently, separating the possible from the impossible part

*Bierens de Haan gives, in such a case, $+\infty$ for the correction for discontinuity.

$$\int_0^\infty \frac{x^{p-1}dx}{(a^2-x^2)^{2n}}\Big[(a-x)^{2n}-\cos p\pi(a+x)^{2n}\Big] = -\frac{2}{2n-1}a^{p-1}\infty^{2n-1}\cos p\pi$$

$$-\pi\frac{p-1}{1}\cdot\frac{p-2}{2}\cdot\cdot,\frac{p-2n+1}{2n-1}a^{p-2n}\sin p\pi, \quad (c_1')$$

$$\int_0^\infty \frac{x^{p-1}dx}{(a-x)^{2n}} = \frac{2}{2n-1}a^{p-1}\infty^{2n-1}-\pi\frac{p-1}{1}\cdot\frac{p-2}{2}\cdots\frac{p-2n+1}{2n-1}a^{p-2n}\cot p\pi$$

$$=\infty. \quad (c_1')$$

The integral (c_1') is likewise ∞ unless $p=\frac{3}{2}$ when we have

$$\int_0^\infty \frac{x^{1/2}dx}{(a+x)^{2n}} = \pi\cdot\frac{1}{2}\cdot\frac{-1}{4}\cdot\frac{-3}{6}\cdots\frac{3-4n+2}{4n-2}a^{\frac{3}{2}-2n}$$

$$=\pi a^{\frac{3}{2}-2n}\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{5}{8}\cdots\frac{4n-5}{4n-2}.$$

Eliminating the integral $\int_0^\infty \frac{x^{p-1}dx}{(a-x)^{2n}}$ we obtain

$$\int_0^\infty \frac{x^{p-1}\,dx}{(a+x)^{2n}} = -\frac{\pi a^{p-2n}}{\sin p\pi}\cdot\frac{p-1}{1}\cdot\frac{p-2}{2}\cdots\frac{p-2n+1}{2n-1}. \quad (c_1'')$$

In Γ functions we have

$$\int_0^\infty \frac{x^{p-1}\,dx}{(1+x)^{2n}} = \frac{\Gamma(p)\Gamma(2n-p)}{\Gamma(2n)} = \Gamma(p)\Gamma(1-p)\frac{2n-p-1}{2n-1}\cdot\frac{2n-p-2}{2n-2}$$

$$\cdots\frac{p-2}{2}\cdot\frac{p-1}{1}.$$

But
$$\int_0^\infty \frac{x^{p-1}\,dx}{(a+x)^{2n}} = a^{p-2n}\int_0^\infty \frac{x^{p-1}\,dx}{(1+x)^{2n}}$$

and by the second theorem of Γ functions

$\Gamma(p)\Gamma(1-p)=\frac{\pi}{\sin p\pi}$, consequently (c_1'') proves correct.

By a similar transformation we have from (c_2)

$$\int_0^\infty \frac{x^{p-1}\,dx}{(a^2-x^2)^{2n+1}}\Big[(a-x)^{2n+1}-e^{p\pi i}(a+x)^{2n+1}\Big] = \frac{2(p-1)}{2n-1}a^{p-2}e^{p\pi i}\infty^{2n-1}$$

$$-\pi i a^{p-2n-1}e^{p\pi i}\frac{p-1}{1}\cdot\frac{p-2}{2}\cdots\frac{p-2n}{2n},$$

and, separating possible from impossible parts,

$$\int_0^\infty \frac{x^{p-1}\,dx}{(a^2-x^2)^{2n+1}}\Big[(a-x)^{2n+1}-\cos p\pi(a+x)^{2n+1}\Big] = \frac{2(p-1)}{2n-1}a^{p-2}\cos p\pi\infty^{2n-1}$$

$$+\pi a^{p-2n-1}\sin p\pi\frac{p-1}{1}\cdot\frac{p-2}{2}\cdots\frac{p-2n}{2n}, \quad (c_2')$$

$$\int_0^{\varpi}\frac{x^{p-1}\,dx}{(a-x)^{2n+1}}=-\frac{2(p-1)}{2n-1}a^{p-2}\infty^{2n-1}$$
$$+\pi a^{p-2n-1}\cot p\pi\frac{p-1}{1}.\frac{p-2}{2}\dots\frac{p-2n}{2n}\quad(c_2')$$
$$=\varpi \text{ if } p>1$$
$$=0 \text{ if } p=1.\quad(c_2'')$$

Eliminating the latter integral we have also

$$\int_0^{\varpi}\frac{x^{p-1}\,dx}{(a+x)^{2n+1}}=\frac{\pi}{\sin p\pi}a^{p-2n+1}\frac{p-1}{1}.\frac{p-2}{2}\dots\frac{p-2n}{2n}.\quad(c_2'')$$

This and formula (c_1'') are comprised in the following

$$\int_0^{\varpi}\frac{x^{p-1}dx}{(a+x)^m}=\frac{\pi}{\sin p\pi}a^{p-m}\frac{p-1}{1}.\frac{p-2}{2}\dots\frac{p-m+1}{m-1}.\quad(c'')$$

Finally we have from (c_3)

$$\int_0^{\varpi}\frac{x^{p-1}dx}{a^2-x^2}\left[a-x-e^{p\pi i}(a+x)\right]=-\pi i a^{p-1}e^{p\pi i},$$

$$\therefore\quad\int_0^{\varpi}\frac{x^{p-1}dx}{a+x}-\cos p\pi\int_0^{\varpi}\frac{x^{p-1}dx}{a-x}=\pi a^{p-1}\sin p\pi,\quad(c'_3)$$

$$\int_0^{\varpi}\frac{x^{p-1}dx}{a-x}=\pi a^{p-1}\cot p\pi.\quad(c'_3)$$

Eliminating $\int_0^{\varpi}\frac{x^{p-1}dx}{a-x}$ we have

$$\int_0^{\varpi}\frac{x^{p-1}dx}{a+x}=\frac{\pi a^{p-1}}{\sin p\pi}.\quad(c_3'')$$

Ex. 2. Required $\int_{-\varpi}^{\varpi}\frac{\varphi(x)dx}{\pm a^2+2bx+x^2}$.

We suppose

$$\frac{\varphi(\pm\varpi+yi)}{\pm a^2+2b(\pm\varpi+yi)+(\pm\varpi+yi)^2}=\frac{\varphi(x+\varpi i)}{\pm a^2+2b(x+\varpi i)+(x+\varpi i)^2}=0,$$

also that $\varphi(z)$ does not become infinite for any value between $\varpi+\varpi i$ and $-\varpi+0i$. The element function can become infinite only if

$$\pm a^2+2bz+z^2=0,$$

viz., if
$$z_1=-b+\sqrt{(b^2\mp a^2)},$$
$$z_2=-b-\sqrt{(b^2\mp a^2)}.$$

1. For the upper sign let $b<a$, then

$$z_1=-b+i\sqrt{(a^2-b^2)}$$

is the only value of z coming into consideration for which $\frac{\varphi(z)}{a^2+2bz+z^2}=\varpi$, because $y_1=+\sqrt{(a^2-b^2)}$ is comprised between the limits of the y-integration, viz., ϖ and 0. We have then, by (12),

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{a^2+2bx+x^2} = \Delta_{-b+i\sqrt{(a^2-b^2)}}.$$

To obtain this correction we have to use (10) which becomes here, since $m = 1$,

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{a^2+2bx+x^2} = \Delta_{-b+i\sqrt{(a^2-b^2)}} = 2\pi i\psi[-b+i\sqrt{(a^2-b^2)}]$$

$$= 2\pi i\left[\frac{z+b-i\sqrt{(a^2-b^2)}}{a^2+2bz+z^2}\varphi(z)\right]_{z=-b+i\sqrt{(a^2-b^2)}}$$

$$= 2\pi i.\frac{\varphi(-b+i\sqrt{(a^2-b^2)})}{2i\sqrt{(a^2-b^2)}}$$

$$= \frac{\pi}{\sqrt{(a^2-b^2)}}\varphi[-b+i\sqrt{(a^2-b^2)}]. \qquad (d)$$

2. Let for the upper sign $b = a$ then the proposed integral becomes

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{(a+x)^2} = \infty \text{ [by } (a_1)]. \qquad (d')$$

3. If for the upper sign $b > a$ we have two unequal real roots. The same is the case for the lower sign whatever the value of a or b. We have then two discontinuous elements, viz., if

$$z_1 = -b+\sqrt{(b^2\mp a^2)}+0.i,$$
$$z_2 = -b-\sqrt{(b^2\mp a^2)}+0.i,$$

because x_1 as well as x_2 are comprised within the limits $\pm \infty$ of the x-integration and $y_1 = y_2 = 0 =$ lower limit of the y integration. We have then to use formula $(10'_3)$ and obtain

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{\pm a^2 + 2bx + x^2} = \pi i\left[\frac{\varphi[-b+\sqrt{(b^2\mp a^2)}]}{2\sqrt{(b^2\mp a^2)}} + \frac{\varphi[-b-\sqrt{(b^2\mp a^2)}]}{-2\sqrt{(b^2\mp a^2)}}\right]$$

$$= \frac{\pi i}{2\sqrt{(b^2\mp a^2)}}\Big[\varphi[-b+\sqrt{(b^2\mp a^2)}]-\varphi(-b-\sqrt{(b^2\mp a^2)})\Big]. \qquad (d'')$$

Corollaries. Let $b = 0$ in (d), then

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{a^2+x^2} = \frac{\pi}{a}\varphi(ai); \qquad (d_a)$$

also let $b = 0$ in (d'') for the lower sign only, then

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{-a^2+x^2} = \frac{\pi i}{2a}\Big[\varphi(a)-\varphi(-a)\Big]; \qquad (d''_a)$$

and if we put $a = 0$ in (d'') we have

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{x(2b+x)} = \frac{\pi i}{2b}\Big[\varphi(0)-\varphi(-2b)\Big]. \qquad (d''_b)$$

These formulæ may be used for any special form of $\varphi(x)$ and after properly separating the real from the imaginary we obtain a great variety of integrals as in Ex. 1.

§ 5.

Another useful formula is obtained by assuming in (6) $z_n = \varpi = y_n$ and $x_0 = 0 = y_0$. We have then

$$\int_0^\infty dx[f(x+\varpi i)-f(x+0.i)] = i\int_0^\infty dy[(\varpi+yi)-f(0+yi)]-\Delta,$$

and if

$$f(x+\varpi i) = f(\varpi+yi) = 0, \tag{11'}$$

$$\int_0^\infty dx\, f(x) = i\int_0^\infty dy\, f(yi) + \Delta. \tag{12'}$$

Since both integrals have the same range of integration we can write this equation thus:

$$\int_0^\infty dx\,[f(x) - i f(xi)] = \Delta. \tag{12$'_1$}$$

If the element function is par we have also

$$\int_0^\infty dx\,[f(x^2) - if(-x^2)] = \Delta, \tag{12$'_2$}$$

and if it is par par we have

$$\int_0^\infty dx\,[f(x^4) - i f(x^4)] = \Delta,$$

or

$$\int_0^\varpi dx\, f(x^4) = \frac{\Delta}{1-i} = \frac{\Delta}{2}\left(1+i\right). \tag{12$'_3$}$$

If the element function is of the form $x^{p-1}f(x^4)$, where p is any number between such limits as condition (11′) requires, we have

$$\int_0^\infty dx\,[x^{p-1}f(x^4) - i^p x^{p-1}f(x^4)] = \Delta,$$

or

$$\int_0^\varpi x^{p-1}dx\, f(x^4) = \frac{\Delta}{1-i^p} = \frac{\Delta}{1-e^{\frac{1}{2}p\pi i}} = -\frac{\Delta e^{-(p+2)\frac{1}{4}\pi i}}{2\sin\frac{1}{4}p\pi} \tag{12$'_4$}$$

$$= \frac{\Delta e^{(2-p)\frac{1}{4}\pi i}}{2\sin\frac{1}{4}p\pi}.$$

This latter formula reduces to $(12'_3)$ if $p=1$.

Ex. 4. Evaluate $\int_0^\varpi \frac{dx}{a^4-x^4}\varphi(x^4)$ if $\varphi(x^4)$ does not become infinite for any value of x between the limits and $\frac{\varphi(x^4)}{a^4-x^4}$ satisfies (11′).

The roots of the equation

$$a^4 - z^4 = 0$$

are

$$z_1 = ae^0\ ; \qquad z_3 = ae^{\pi i}\ ;$$

$$z_2 = ae^{\frac{1}{2}\pi i}; \qquad z_4 = ae^{\frac{1}{2}3\pi i}.$$

Only the roots $z_1 = a(1+0.i)$; $z_2 = a(0+i)$ need to be considered since x and y must be positive. We have by $(10'_3)$

$$\Delta_{a(1+0.i)} = \pi i\,\varphi(a^4)\left[\frac{z-a}{a^4-z^4}\right]_{z=a} = -\frac{\pi i}{4a^3}\varphi(a^4),$$

and by $(10''_3)$

$$\Delta_{a(0+i)} = \pi i\,\varphi(a^4)\left[\frac{z-ai}{a^4-z^4}\right]_{z=ai} = -\frac{\pi i}{4a^3 i^3}\varphi(a^4) = \frac{\pi}{4a^3}\varphi(a^4);$$

$$\therefore\ \Delta = \frac{\pi}{4a^3}\,\varphi(a^4)\,(1-i).$$

Hence by ($12'_3$)

$$\int_0^\infty \frac{dx}{a^4-x^4}\,\varphi(x^4) = \frac{\pi}{4a^3}\,\varphi(a^4). \qquad (e)$$

Let $\varphi(x^4) = e^{-bx^4}$ then $\dfrac{e^{-b(\infty+yi)4}}{a^4-(\infty+yi)^4} = \dfrac{e^{-b(x+\infty i)4}}{a^4-(x+\infty i)^4} = 0$, and therefore

$$\int_0^\infty \frac{dx}{a^4-x^4}\,e^{-bx^4} = \frac{\pi}{4a^3}\,e^{-a^4b}. \qquad (e')$$

Ex. 5. Evaluate $u = \displaystyle\int_0^\infty dx\frac{x^{p-1}}{a+x}\varphi(x)$.—Replace x by x^4 then

$$u = 4\int_0^\infty \frac{x^{4p-1}\,dx}{a+x^4}\,\varphi(x^4).$$

This is of the form ($12'_4$) and if $\varphi(z^4)$ does not become infinite between the limits ∞ and 0 of x and y, the points of discontinuity are given by the eq'n

$$a + z^4 = 0,$$

which has the four roots

$$z_1 = a^{\frac{1}{4}}e^{\frac{1}{4}\pi i}; \qquad z_3 = a^{\frac{1}{4}}e^{\frac{5}{4}\pi i};$$

$$z_2 = a^{\frac{1}{4}}e^{\frac{3}{4}\pi i}; \qquad z_4 = a^{\frac{1}{4}}e^{\frac{7}{4}\pi i}.$$

Of these only $z_1 = a^{\frac{1}{4}}(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$ falls within the limits, since x as well as y must be positive. By formula (10) we have then

$$\Delta_{a^{\frac{1}{4}}e^{\frac{1}{4}\pi i}} = 2\pi i . a^{p-\frac{1}{4}}e^{(p-\frac{1}{4})\pi i}\varphi(-a)\left[\frac{z-a^{\frac{1}{4}}e^{\frac{1}{4}\pi i}}{a+z^4}\right]_{z=a^{\frac{1}{4}}e^{\frac{1}{4}\pi i}}$$

$$= \tfrac{1}{2}\pi i\,a^{p-1}e^{(p-1)\pi i}\,\varphi(-a).$$

This being used for Δ in ($12'_4$) we obtain

$$u = \int_0^\infty \frac{x^{p-1}dx}{a+x}\,\varphi(x) = 4\int_0^\infty \frac{x^{4p-1}dx}{a+x^4}\,\varphi(x^4) = 4.\tfrac{1}{2}\pi i\,a^{p-1}e^{(p-1)\pi i}\varphi(-a) \times \frac{e^{-(p-\frac{1}{2})\pi i}}{2\sin p\pi}$$

$$= \frac{\pi a^{p-1}}{\sin p\pi}\,\varphi(-a). \qquad (f)$$

Let $\varphi(x) = 1$, then if $2 > p > 1$ ($11'$) is satisfied and we obtain the familiar integral already given, § 4, Ex. 1 and 2,

$$\int_0^\infty \frac{x^{p-1}dx}{a+x} = \frac{\pi a^{p-1}}{\sin p\pi}. \qquad (f')$$

[Mr. Kummell has contributed another § to this paper but for want of suitable type we are not able to insert it at present, but hope to be able to do so before the close of the present volume.]